REGULARIZATION OF MIXED SECOND-ORDER ELLIPTIC PROBLEMS

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ABSTRACT

It is shown that weak solutions of mixed elliptic problems are Hölder continuous of any order less than 1/2 and that they possess higher regularity in non-critical directions.

0. **Introduction and statement of results.** The regularization principle for elliptic equations and boundary value problems roughly states that weak solutions are strong and strong solutions are as smooth as the coefficients of the operators, the domain and the data allow. For example, if A is a linear elliptic operator of order m, A^* its formal adjoint and $u \in L^p(\Omega)$ satisfies $\int uA^*\phi dx = \int f\phi dx$ for all $\phi \in C_0^\infty(\Omega)$ with $f \in L^p(\Omega)$, then u has generalized derivatives up to order m in $L^p(\Omega')$, Ω' a compact subdomain of Ω. Similar (but deeper) results establish strong differentiability up to the boundary of weak solutions of boundary value problems. We mention in particular the works of Agmon [1], Lions-Magenes [6] Schechter [10] and the exposition of Magenes [8]. The regularity principle is known to be valid for local solutions, and this is the approach of [1]. In [6, 10] the treatment is global. Another feature of these works is that, rather than posing the problems in a weak form (as in the example above), the operators are extended (by duality and interpolation) to mappings between suitable function or distribution spaces in Ω or $\partial\Omega$. The regularization is carried out in this framework, and usually appears as invertibility of the extended mappings. This method requires the development of a rather heavy machinery of distribution spaces and their trace (boundary) behaviour. This is, however, a natural functional-analytic approach to boundary value problems and in addition there is an interplay between the theory of elliptic problems and the abstract properties of the distribution spaces.

In this article we regularize (very) weak solutions of mixed second order real elliptic problems (in any dimension). Thus our solutions satisfy a boundary condition which breaks across a smooth $(n-1)$-dimensional submanifold Γ of $\partial\Omega$ (which is smooth and n-dimensional). Due to the local character of the known regularity results, only the behaviour near points of Γ has to be studied. We shall

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use a global approach to prove invertibility (in suitable spaces) of certain canonical mixed problems where $A = -\Delta + \delta^2$, the boundary operators are Dirichlet (the trace) and constant oblique derivatives, the domain is the half space $R_+^{n+1} = \{(t, y, x') \mid t > 0)\}$ and the break manifold Γ is the plane $t = 0$, $y = 0$. But in proving regularity for general mixed problems, in particular partial regularity in directions parallel to Γ, we also use local methods close to Agmon's [1].

There is an upper limit to the total regularity one can expect for mixed problems. This is seen in the following example. Let $u(t, y) = \text{Im}(y + it)^{\frac{1}{2}}$ in the half plane $t > 0$. It is harmonic there and

$$\lim_{t \to 0} u(t,y) = 0, \quad y > 0; \quad \lim_{t \to 0} \frac{du}{dt}(t,y) = 0, \quad y < 0.$$

The data of this mixed Dirichlet-Neumann problem is as smooth as one can wish—identically zero. Still, Hölder continuity of order $\frac{1}{2}$ is the most one can get here.

Our results are summarized as follows. Let A be a real elliptic operator in $\Omega \subset R^{n+1}$. Let the (real) boundary operator "cover" A at each point of $\partial\Omega$. Assume that the coefficients of the operators, $\partial\Omega$ and Γ (the break manifold) are smooth (C^3 is enough), and let the data (for a nonhomogeneous problem) be reasonably smooth. Let $u \in L^p$, $p > 4$ or let $u \in H^{1,q}$, $q > 1$ be a weak local solution of the mixed problem near $X_0 \in \Gamma$. Then u is (near X_0) in the space $H^{s,p}$ for every $s < \frac{1}{2} + \frac{2}{p}$ (hence if p is large enough u is α-Hölder continuous, for any $\alpha < \frac{1}{2}$). Moreover, u has partial regularity in directions parallel to Γ, i.e., first derivatives of u in such directions belong to L^p. These are the directions orthogonal to t and y after a neighborhood of X_0 in Ω is transformed on a half-spherical neighborhood of the origin $\{X = (t, y, x') \mid t \geq 0, |X| < \delta\}$ and the break manifold goes over to the plane $t = 0$, $y = 0$.

Higher regularity in the directions parallel to Γ is obtained if the operator have constant coefficients. We believe this is true in general and the barrier of regularization exists only in the two "critical" directions. In these two directions one can regularize u up to order $\frac{1}{2} - \varepsilon$ in terms of Hölder continuity. Thus our results are optimal—except for the possibility that the ε can be dropped, but this is not likely.

Hölder continuity at some order α (which is difficult to estimate) for solutions in $H^{1,2}$ also follows from general results of Stampacchia [14] about elliptic problems with various discontinuities. Continuity for some special, mixed boundary problems was obtained by Fichera [4]. The general method of Stampacchia [14] is closer to potential theory. Our methods are specialized to mixed problems, and essentially use reduction to pure elliptic problems. The singular integral equations involved in this reduction were studied in [13] (see Footnote in Section 3), even for the vectorial case which arises in the treatment of higher order mixed problems. Some of these problems are amenable (albeit technical difficulties) to our method of regularization.

The plan of the paper is the following:

1. Notations, spaces of functions and distributions.
2. Pure problems for $-\Delta + \delta^2$ in a half space.
3. Mixed problems for $-\Delta + \delta^2$ in a half space.
4. Perturbed problems and local fundamental solution.
5. Regularity in the general case.

I am indebted to Professor S. Agmon for useful conversations concerning this paper.

1. **Notations, spaces of functions and distributions.** Let $x = (x_1, \cdots, x_n)$ represent a point in R_n, $\xi = (\xi_1, \cdots, \xi_n)$ a point in the dual space. We also use the notation $x_1 = y$, $x = (y, x')$, $\xi_1 = \eta$, $\xi = (\eta, \xi')$, $R^n_+ = \{(y,x') \mid y > 0\}$ and $R^n_- = \{(y,x') \mid y < 0\}$. The spaces $C^k(\Omega)$, $C_0^\infty(\Omega)$ have their customary meaning. Let $H^{s,p} = H^{s,p}(R^n)$ be the space of distributions u for which

$$\| u \|_{s,p} = \| F^{-1}(1 + |\xi|^2)^{s/2} Fu \|_{L^p(R^n)} < \infty, \tag{1.1}$$

here $(Fu)(\xi) = \int u(x) e^{ix \cdot \xi} dx$ is the Fourier transform. We assume $1 < p < \infty$. For an integer $k \geqq 0$, $H^{k,p}$ consists of all functions with distribution derivatives up to order k belonging to L^p. Alternatively, $H^{s,p}$ is obtained as the closure of $C_0^\infty(R^n)$ under the norm (1.1). For basic properties of the spaces $H^{s,p}$ (and $W^{s,p}$ used later) we refer the reader to [6, 7, 8, 10]. We describe here some properties which are particularly relevant later.

Let Ω be a domain, $\bar{\Omega}$ its closure. We set

$$H^{s,p}_{\bar{\Omega}} = \{ u \in H^{s,p} \mid \text{support } u \subseteq \bar{\Omega} \}. \tag{1.2}$$

For $\Omega = R^{n+1}_\pm$, the space (1.2) is denoted $H^{s,p}_\pm$. If $\partial\Omega$ is smooth, non self-intersecting (as we shall always assume), $H^{s,p}_{\bar{\Omega}}$ is the closure of $C_0^\infty(\Omega)$ in $H^{s,p}$. Let

$$H^{s,p}(\Omega) = H^{s,p} / H^{s,p}_{\Omega'}\ , \ \Omega' \text{ is the closed complement of } \Omega. \tag{1.3}$$

This is the space of all restrictions of $u \in H^{s,p}$ to Ω, equipped with the quotient norm. For $\Omega = R^n_\pm$, $Y_\pm$: $H^{s,p} \to H^{s,p}(R^n_\pm)$ is the projection onto the quotient space, and $\| Y_\pm u \|_{s,p}$ denotes the quotient form. Thus $Y_+ u$ is the class $u - H^{s,p}_-$, and is represented by u or any other element in this class. For $s = 0$ we can identify $L^p(R^n_\pm)$ with $L^p_\pm$ and $Y_\pm$ with multiplication by the characteristic function of $R^n_\pm$. This is actually true for $-1 + 1/p < s < 1/p$ (see Remark 1.5). For real r let

$$J^r_\delta = F^{-1}(\delta^2 + |\xi|^2)^{r/2} F, \quad \delta > 0. \tag{1.4}$$

Then for each δ, $\| J^r_\delta u \|_{L^p}$ is a norm equivalent to $\| u \|_{r,p}$ (equal if $\delta = 1$). More generally, J^r_δ maps $H^{s,p}$ onto $H^{s-r,p}$. Let

$$J^r_{\pm,\delta} = F^{-1}[\eta \pm i(|\xi'|^2 + \delta^2)^{\frac{1}{2}}]^r F. \tag{1.5}$$

(Usually we omit the subscript δ.) Both operators are invertible, and map $H^{s,p}$ onto $H^{s-r,p}$ so the norms $\| J^r_{\pm,\delta} u \|_{L^p}$ are again equivalent to $\| u \|_{s,p}$. However J^r_- also maps $H^{s,p}_-$ onto $H^{s-r,p}_-$, because the factor within brackets in (1.5) is analytic for $\mathrm{Im}\,\eta < 0$. Hence $J^r_-(u + H^{s,p}_-) = J^r_- u + H^{s-r,p}_-$, i.e. $J^r_- Y_+ u = Y_+ J^r_- u$. J^r_+ acts in the dual fashion (in particular J^r_+ and Y_- commute) and we have the norm relations

$$\| Y_\pm u \|_{s,p} \sim \| Y_\pm J^r_\mp u \|_{s-r,p} (\sim \| Y^s_\pm J_\mp u \|_{L^p} \quad \text{if} \quad s = r). \tag{1.6}$$

A differentiation of order k maps $H^{s,p}$ into $H^{s-k,p}$ and since it preserves supports, it acts similarly in $H^{s,p}_{\bar{\Omega}}$, $H^{s,p}(\Omega)$. Multiplication by a function of $C^k(\Omega)$ also preserves supports and maps $H^{s,p}(\Omega)$ into itself if $|s| \leqq k$. For $s < 0$, this follows from the reflexitivity of our spaces, the conjugate spaces being

$$(H^{s,p})' = H^{-s,p'}; \quad (H^{s,p}(\Omega))' = H^{-s,p'}_{\bar{\Omega}}, \quad 1/p + 1/p' = 1. \tag{1.7}$$

LEMMA 1.1 [11, 13]. *The map $u \to (Y_- u, Y_+ u)$ of $H^{s,p}$ into $H^{s,p}(R^n_-) \times H^{s,p}(R^n_+)$ is invertible if and only if $-1 + 1/p < s < 1/p$. If $s > 1/p$ but $s - 1/p$ is not an integer, then the map is* 1-1 *and has a closed range.*

We consider a more general operator, with domain and range as above

$$\tilde{M}: u \to (Y_- u, Y_+ F^{-1} M(\xi) F u). \tag{1.8}$$

We assume that $M = F^{-1}M(\xi)F$ is bounded and invertible in $H^{s,p}$. Expressing the norms in $H^{s,p}(R^n_\pm)$ by means of L^p norms, using (1.6), it is shown [13, Theorem 6.1] that

LEMMA 1.2. *$\tilde{M}$ is invertible* (1-1, *onto, etc.*) *for* (s,p) *if and only if $\tilde{M}_s$ is invertible* (1-1, *onto, etc.*) *for* $(0,p)$ *where $\tilde{M}_s$ is associated with the factor*

$$M_s(\xi) = M(\xi) \left(\frac{\eta - i(|\xi'|^2 + \delta^2)^{\frac{1}{2}}}{\eta + i(|\xi'|^2 + \delta^2)^{\frac{1}{2}}} \right)^s, \qquad \delta > 0 \textit{ fixed}. \tag{1.9}$$

This in turn is equivalent to the operator

$$Y_+ v \to Y_+ F^{-1} M_s(\xi) F Y_+ v \textit{ acting in } L^p(R^n_+) \tag{1.10}$$

being invertible (1-1, *onto, etc.*).

To describe traces of $H^{s,p}(\Omega)$ in $\partial\Omega$, we need the spaces $W^{s,p}$ (which coincide with $H^{s,p}$ for integral s). There are several equivalent definitions, and we refer the reader to the literature (e.g. [6], [7], [8], [15]). The spaces $W^{s,p}_\pm$, $W^{s,p}(R^n_\pm)$, $W^{s,p}_{\bar{\Omega}}$, $W^{s,p}(\Omega)$, and the maps $Y_\pm$ are defined as in the $H^{s,p}$ case. The relations (1.7) serve as definitions of the spaces for negative s. Differentiations and multiplications (in general, linear partial differential operators) again act in the same way, and so do J^r, $J^r_\pm$ but only if r is an integer. Lemma 1.1 is also valid [6, IV]. We shall use $\| \phi \|_{s,p}$, $\| Y_\pm \phi \|_{s,p}$ etc. to denote $W^{s,p}$ norms as well, usually for functions denoted by Greek letters, but no ambiguity should arise, especially as $W^{s,p}$ norms will be used for traces on $\partial\Omega$ or $R^n = \partial R^{n+1}_+$, and $H^{s,p}$ norms will be used in Ω or R^{n+1}_+.

REMARK 1.3. The reduction of $\| Y_\pm u \|_{s,p}$ to L^p norms in (1.6) is not true any more for $W^{s,p}$, unless s is an integer. But the following is still true:

If the operator (1.8) is invertible in the $H^{s,p}$ scale for $s_1 < s < s_2$, then it is invertible in the $W^{s,p}$ scale for $s_1 < s < s_2$. This follows from the fact that $W^{s,p}$ is obtainable from $H^{s_1,p}$ and $H^{s_2,p}$ by interpolation [7].

THEOREM 1.4 (*The trace theorem*). *Let $k > j \geqq 0$. Then*

$$\gamma_j u(t,x) = (D_t^j u)(0,x), \quad \left(D_t = \frac{\partial}{\partial t}\right),$$

defined originally for C^k functions, extends to a bounded map

$$\gamma_j : H^{k+\theta,p}(R^{n+1}_+) \text{ onto } W^{k-j+\theta-1/p,p}(R^n), \tag{1.11}$$

θ is always assumed to satisfy $-1 + 1/p < \theta < 1/p$. γ_0 is also denoted by γ and called the trace operator. The same result holds for $W^{k+\theta,p}$ instead of $H^{k+\theta,p}$ (the trace space is the same in both cases). If Ω has a smooth boundary the traces of the normal derivatives $\gamma_j u = \partial^j u/\partial n^j |_{\partial\Omega}$ behave in the same way.

In addition to the works we have already referred to, we mention the simple and direct proof of Stein [15].

REMARK 1.5 (applies also to $W^{s,p}$ spaces). For $k \geqq 1$, the subspace of $H^{k+\theta,p}(\Omega)$ determined by the conditions $\gamma_j u = 0$, $0 \leqq j \leqq k-1$, coincides with $H^{k+\theta,p}_{\bar\Omega}$. Thus for $s > 1/p$, $u \in H^{s,p}(\Omega) \cap H^{1+\theta,p}_{\bar\Omega}$ is equivalent to $\gamma_0 u = 0$. For $-1 + 1/p < \theta < 1/p$, $H^{\theta,p}(\Omega)$ can be identified with $H^{\theta,p}_{\bar\Omega}$, $C_0^\infty(\Omega)$ being dense in both spaces. A function $u \in H^{\theta,p}(\Omega)$ can be extended as 0 in Ω' or be pieced together with a function in $H^{\theta,p}(\Omega')$ to obtain a function of $H^{\theta,p}$ (on the whole euclidean space). This is the reason why the map on Lemma 1.1 is onto for $-1 + 1/p < s < 1/p$. For $s = k + \theta$, $k \geqq 1$, the "piecing" requires compatibility conditions (e.g., $\gamma_1 u = \cdots = \gamma_{k-1} u = 0$ if we extend u as 0).

2. **Pure problems for $-\Delta + \delta^2$ in a half-space.** Let Δ denote the Laplacian with respect to all the variables $(t,x) \in R^{n+1}$. Consider the elliptic operator

$$Au = (-\Delta + \delta^2)u, \; \delta > 0 \text{ a fixed parameter.} \tag{2.1}$$

The characteristic form of A admits the decomposition

$$\tau^2 + |\xi|^2 + \delta^2 = [\tau - i(|\xi|^2 + \delta^2)^{\frac12}] \cdot [\tau + i(|\xi|^2 + \delta^2)^{\frac12}].$$

We now use $T_{\pm}$ to denote the projections of $H^{s,p}$ onto $H^{s,p}(R^{n+1}_{\pm})$. Then the operators

$$A_{\pm} = F^{-1}[\tau \pm i(|\xi|^2 + \delta^2)^{\frac{1}{2}}]F \tag{2.2}$$

take the place of $J_{\pm}$ of (1.5). In particular we have $A^r_-T_+ = T_+A^r_-$.

We shall solve boundary value problems for A in the half space $R^{n+1}_+ = \{(t,x) \mid t > 0\}$, in the framework of the function spaces $H^{s,p}$. The statements and the formulas remain true, however, for $W^{s,p}$ as well. We refer to Arkeryd [3] for a similar approach.

The *homogeneous Dirichlet problem* (HDP) in the space $H^{1+\theta,p}(R^{n+1}_+)$ is the following:

(2.3) *Given* $f \in H^{-1+\theta,p}(R^{n+1})$, *find* $u \in H^{1+\theta,p}_+$ *satisfying* $T_+Au = T_+f$.

The condition $u \in H^{1+\theta,p}_+$ means that $\gamma_0 u = 0$.

THEOREM 2.1. *The unique solution of the* HDP (2.3) *is*

$$u = A_+^{-1}T_+A_-^{-1}f \tag{2.4}$$

Moreover, if $f \in H^{k+\theta,p}$, $k \geqq 0$, *then* $T_+u \in H^{k+2+\theta,p}(R^{n+1}_+)$.

Proof. Let u be a solution of (2.3). Then $T_+A_-A_+u = T_+f$. Operating with $A_+^{-1}A_-^{-1}$ and commuting A_-^{-1} with T_+ we get

$$A_+^{-1}T_+A_+u = A_+^{-1}T_+A_-^{-1}f \tag{2.5}$$

since $u \in H^{1+\theta,p}_+$, $A_+u \in H^{\theta,p}_+$, on which space T_+ acts by multiplication with the characteristic function of R^{n+1}_+ (recall that $-1 + 1/p < \theta < 1/p$). Thus $T_+A_+u = A_+u$ and $A_+^{-1}T_+A_+u = u$. Comparing with (2.5), we see that a solution must have the form (2.4). Conversely, if u is given by (2.4), then $T_+Au = T_+f$. Also $T_+A_-^{-1}f$ is in $H^{\theta,p}_+$ and A_+^{-1} takes it to $H^{1+\theta,p}_+$.

If $f \in H^{k+\theta,p}$, $k \geqq 0$, we consider $v = A_-^{-k-1}A_+^{-1}T_+A_-^kf$. This is $A_+^{-1}T_+A_-^{-1}f$ modulo a function supported in $\bar{R}^{n+1}_-$, hence $T_+u = T_+v$. But A_-^kf and $T_+A_-^kf$ is in $H^{\theta,p}$, and this is taken to $H^{k+2+\theta,p}$ by $A_-^{-k-1}A_+^{-1}$. (Clearly $\gamma_0 v = \gamma_0 u = 0$).

The non homogeneous Dirichlet problem (NDP) *can be posed as*

$$T_+Au = T_+f \text{ and } u - u_0 \in H^{1+\theta,p}_+ , \tag{2.6}$$

where u_0 is given in $H^{1+\theta,p}$. The second condition means that $\gamma u = \gamma u_0$. Having solved the HDP, we may assume here that $f \neq 0$. We set $u = v + u_0$. Then v should satisfy $T_+Av = -T_+Au_0$ and $v \in H^{1+\theta,p}_+$. Hence $v = A_+^{-1}T_+A_-^{-1}Au_0$ $= -A_+^{-1}T_+A_+u_0$ and

$$u = u_0 - A_+^{-1}T_+A_+u_0. \tag{2.7}$$

In case $u_0 \in H^{k+2+\theta,p}$, $k \geqq 0$, then T_+v, hence also T_+u belong to $H^{k+2+\theta,p}(R_+^{n+1})$.

In view of Theorem 1.4, giving u_0 is equivalent to giving the boundary condition $\gamma u = \gamma u_0 = \phi$, $\phi \in W^{k+2+\theta-1/p,p}$. Summarizing, we have

THEOREM 2.3. *The map $T_+u \to (T_+Au, \gamma u)$ is an (onto) isomorphism*

$$H^{k+\theta,p}(R_+^{n+1}) \to H^{k-2+\theta,p}(R_+^{n+1}) \times W^{k+\theta-1/p,p}(R^n), \quad k \geqq 1.$$

We wish to consider other boundary problems, e.g. Neumann's:

$$T_+Au = T_+f, \; \gamma_1 u = \frac{d}{dt}u\Big|_{t=0} = \psi, \tag{2.8}$$

in $H^{k+\theta,p}$. If $k = 0,1$ we have to show first that $\gamma_1 u$ makes sense. To this end, we define

$$H_A^{s,p}(\Omega) = \{u \mid u \in H^{s,p}(\Omega) \;\; and \;\; Au \in L^p(\Omega)\}. \tag{2.9}$$

THEOREM 2.4. *The maps $\gamma_0 : u \to u|_{t=0}$, $\gamma_1 : u \to D_t u|_{t=0}$ have (unique) extensions*

$$\gamma_0 : H_A^{\theta,p}(R_+^{n+1}) \xrightarrow{onto} W^{\theta-1/p,p}(R^n) \tag{2.10}$$

$$\gamma_1 : H_A^{k+\theta,p}(R_+^{n+1}) \xrightarrow{onto} W^{k-1+\theta-1/p,p}(R^n), \quad k = 0,1. \tag{2.11}$$

If $T_+Au = T_+f$, then γ_0 and γ_1 are related by

$$\gamma_1 u = J_\delta \gamma_0 u - i\gamma_0 A_-^{-1} f \tag{2.12}$$

Proof. We start with (2.11) for $k=1$. We have $T_+A_+u = T_+A_-^{-1}f \in H^{1,p}(R_+^{n+1})$. By (2.2) and (1.4)

$$A_\pm = i\left[\frac{d}{dt} \pm F^{-1}(|\xi|^2 + \delta^2)^{1/2}F\right] = -i\left(\frac{d}{dt} \mp J_\delta\right). \tag{2.13}$$

Hence for smooth u

$$T_+\frac{du}{dt} = T_+J_\delta u + iT_+A^{-1}f. \tag{2.14}$$

Now $\gamma_0 A^{-1} f \in W^{1-1/p,p}$. Also, γ_0 and J_δ commute, J_δ being a purely tangential operator (of order 1) and $\gamma_0 J_\delta u = J_\delta(\gamma_0 u) \in W^{\theta-1/p,p}$. Since for any v $\gamma_0 T_+ v = \gamma_0 v$, we see that the trace of the right hand side of (2.14) gives the required extension. If $f = 0$ then $\gamma_1 u = J_\delta \gamma_0 u$. By Theorem 2.3 $\{\gamma_0 u \mid T_+Au = 0, \; u \in H^{1+\theta,p}\}$ exhausts $W^{1+\theta-1/p,p}$, which space is mapped *onto* $W^{\theta-1/p,p}$ by J_δ.

For (2.10), we set $v = A_-^{-1}u \in H^{1+\theta,p}$. Then $T_+Av = T_+A^{-1}f$, and by the part already proved, $\gamma_1 v \in W^{\theta-1/p,p}$. Now $u = A_-v = i(D_t v + J_\delta v)$. Taking traces, we see that $\gamma_0 u = -i\gamma v + J_\delta \gamma_0 v \in W^{\theta-1/p,p}$ is the required extension. For $f = 0$

we add $T_+A_+v = 0$ to $T_+A_-v = T_+u$, and obtain $-2iT_+D_tv = T_+u$, hence $\gamma_0 u = -2i\gamma_1 v$ and these exhaust $W^{\theta-1/p,p}$ by the part already proved.

Finally, once $\gamma_0 u$ is defined for $H^{\theta,p}$, we repeat the argument of the first part to prove (2.11) and (2.12) for $k=0$.

To prove that the extensions are unique, it suffices to show that the smooth functions are dense in $H_A^{k+\theta,p}(R_+^{n+1})$. We postpone this to Remark 2.6 at the end of this section. We note that (2.10) and Theorem 2.3 already imply

THEOREM 2.3 BIS. *The map $T_+u \to (T_+Au, \gamma_0 u)$ of $H_A^{\theta,p}$ onto $L^p \times W^{\theta-1/p,p}$ is an isomorphism.*

Now we solve the Neumann problem (2.8). For u to be in $H^{k+\theta,p}$, f should be in $H^{k-2+\theta,p}$ and ψ in $W^{k-1+\theta-1/p,p}$. Having solved the NDP, we may assume $f=0$. Then $\gamma_1 u = J_\delta \gamma_0 u$ (by (2.12), which is obviously true for every $k \geqq 0$). Given ψ, we set $J_\delta^{-1}\psi = \phi$ and solve the Dirichlet problem $Au=0$, $\gamma_0 u = \phi$. Its solution $u \in H^{k+\theta,p}$ (or $H_A^{k+\theta,p}$ if $k=0,1$) is clearly the unique solution of the Neumann problem: $T_+Au=0$, $\gamma_1 u = \psi$.

For the purpose of the next section, however, it is convenient to reduce the Neumann problem to a *modified Dirichlet problem* (MDP), which assigns the 1st order tangential operator $J_+\gamma_0 u$ instead of $\gamma_0 u$. (The two forms are equivalent since J_+ is invertible.) If $T_+Au=0$ then

(2.15) $$\gamma_1 u = JJ_+^{-1}(J_+\gamma_0 u),$$

(2.16) $$JJ_+^{-1} = F^{-1}\frac{\eta^2 + |\xi'|^2 + \delta^2}{\eta + i(|\xi'|^2+\delta^2)^{1/2}}F = F^{-1}\left(\frac{\eta - i|\xi'_\delta|}{\eta + i|\xi'_\delta|}\right)^{1/2}F$$

and $|\xi'_\delta| = (|\xi'|^2+\delta^2)^{1/2}$—notice that J and J_+ are actually J_δ and $J_{+,\delta}$ of (1.4) and (1.5), but occasionally we omit the δ to simplify the notation. This operator is invertible in $W^{s,p}$ for any s. Using either reduction, we have

THEOREM 3.5. *The map $T_+u \to (T_+Au, \gamma_1 u)$ is an isomorphism.*

(2.17) $$H^{k+\theta,p}(R_+^{n+1}) \xrightarrow{onto} H^{k-2+\theta,p}(R_+^{n+1}) \times W^{k+\theta-1-1/p,p}(R^n), \quad k \geqq 2$$

(2.18) $$H_A^{k+\theta,p}(R_+^{n+1}) \xrightarrow{onto} L^p(R_+^{n+1}) \times W^{k+\theta-1-1/p,p}(R^n), \quad k=0,1$$

The same is true for the map $T_+u \to (T_+Au, \gamma_0 Bu)$ where

(2.19) $$B = D_t + \sum_{j=1}^{n}\alpha_j D_j, \quad \alpha_j \text{ real}, \quad \left(D_j = \frac{d}{dx_j}\right),$$

is a fixed oblique derivative.

Proof. For the oblique derivative we compute, in analogy to (2.15), (2.16), that for solutions of $T_+Au=0$

$$\gamma_0 Bu = F^{-1}M(\eta,\xi')F(J_+\gamma_0 u) \tag{2.20}$$

where

$$M(\eta,\xi') = \left(\frac{\eta - i|\xi'_\delta|}{\eta + i|\xi'_\delta|}\right)^{1/2}\left[1 - \frac{i\Sigma\alpha_j\xi_j}{\eta^2 + |\xi'|^2 + \delta^2}\right]. \tag{2.21}$$

$M(\eta,\xi')$ never vanishes and (2.20) is invertible in $W^{s,p}$ for any s.

REMARK 2.6. The following results, some of which were obtained here in a special situation, are true in general for a properly elliptic (even higher order) operator with smooth coefficients defined in a smooth domain Ω, provided the Dirichlet problem for A in Ω has a unique solution.

(a) The smooth functions are dense in $H_A^{k+\theta,p}$, $k = 0,1$.

(b) The maps γ_0 and γ_1 (trace of the normal derivative) have unique extensions to $H_A^{k+\theta,p}(\Omega)$ and (2.10–11) is true (with Ω and $\partial\Omega$).

(c) Theorem 2.3 bis for this situation is true.

(d) Let A^* be the formal adjoint of A and B a 1st order differential operator whose principal part at each point of $\partial\Omega$ is an oblique derivative (i.e. never tangent to $\partial\Omega$). The Green's formula

$$\int_\Omega Au\cdot v dx - \int_\Omega u\cdot A^*v dx = \int_{\partial\Omega}\gamma_0[Bu\cdot v - u\cdot B^*v]d\sigma, \tag{2.22}$$

which holds for a suitable B^* and smooth u,v, extends to all $u\in H_A^{k+\theta,p}$. The right hand side becomes $\langle\gamma_0 Bu,\gamma_0 u\rangle - \langle\gamma_0 u,\gamma_0 B^*v\rangle$ where $\langle\,,\rangle$ denotes in each case the duality between the space to which $\gamma_0 u$ or $\gamma_0 Bu$ belong and its adjoint.

These results are proved in [6, V] in the following order (indeed for $W^{s,p}(\Omega)$ but the proofs carry over to $H^{s,p}(\Omega)$ and conversely—our proofs carry to $W^{s,p}(R_+^{n+1})$): First (a) is proved for $k+\theta = 0$ [V, prop. 3.1]. Next (b) is proved for $k+\theta = 0$ [V, Theorem 3.1] by using (a) and the Green's formula, which is thereby extended. The proof of (c) then follows by an interpolation technique using results for $H^{2,p}$ and $H^{0,p}$ (obtained by a duality argument). Finally (a) is proved for general $k+\theta$. This last proof we present for our $H_A^{k+\theta,p}(R_+^{n+1})$, in order to complete the proof of Theorem 2.4 (the uniqueness of γ_0,γ_1).

Let $T_+Au = T_+f$, $u\in H_A^{s,p}(R_+^{n+1})$, $(s = k+\theta)$. There is a unique solution w to the HDP $T_+Aw = T_+f$, $\gamma_0 u = 0$, and (with norms taken in R_+^{n+1})

$$\|w\|_{H_A^{s,p}} \leqq \|T_+w\|_{2,p} + \|T_+f\|_{0,p} \leqq C\|T_+f\|_{0,p} \tag{2.23}$$

set $v = u - w$. Then $T_+Av = 0$, $\gamma_0 v = \gamma_0 u\in W^{s-1/p,p}$. (Using the γ_0 found in Theorem 2.4 for small s) and by Theorem 2.3 bis.

$$\|v\|_{H_A^{s,p}} \leqq C\|\gamma_0 v\|_{s-1/p,p} \tag{2.24}$$

Let now the sequence of smooth f_j converge to f in $L^p(R_+^{n+1})$, and w_j the (smooth) solution of the HDP $T_+w_j = T_+f_j$. Then $w_j - w\to 0$ in $H_A^{s,p}(R_+^{n+1})$ by (2.23).

Also if the sequence of smooth ϕ_j converges to $\gamma_0 v$ in $W^{s-1/p,p}$ and v_j is the (smooth) solution of $T_+Av_j = 0$, $\gamma_0 v_j = \phi_j$, then $v_j - v \to 0$ in $H_A^{s,p}(R_+^{n+1})$ by (2.24). Hence $w_j + v_j \to u$ in that space.

3. **Mixed Problems for** $-\Delta + \delta^2$ **in a half space.** Consider first the mixed Dirichlet-Neumann problem in $R^{n+1} = \{(t,y,x') \mid t > 0\}$:

$$(3.1) \qquad (T_+A, Y_-J_+\gamma_0, Y_+\gamma_1)u = (T_+f, Y_-\phi, Y_+\psi),$$

where we have used the modified Dirichlet condition $Y_-J_+\gamma_0 u = Y_-\phi$ in R^n, which is equivalent to assigning $Y_-\gamma_0 u$ in R^n since $J_+ : W^{s,p}(R_-^n) \to W^{s-1,p}(R_-^n)$ is an (onto) isomorphism.

From Theorems 1.4, 2.4 we deduce that the following maps are continuous.

$$(3.2) \quad H_A^{k+\theta,p} \in T_+u \to (T_+Au, Y_-J_+\gamma_0 u, Y_+\gamma_1 u) \in H^{k',p} \times \Pi_\pm W^{k+\theta-1-1/p,p}(R_\pm^n)$$

(the H spaces are on R_+^{n+1}). Here $k' = 0$ if $k = 0, 1$, while $k' = k + \theta - 2$ and the subscript A is superfluous if $k \geqq 2$. We shall find the values of $(k+\theta, p)$ for which (3.2) is invertible.

Subtracting a solution of the *HDP* $Y_+Au = Y_+f$, we may assume $f = 0$. Then using the relation (2.15), we find for a function u satisfying $T_+Au = 0$ the following relation between its mixed data and modified Dirichlet data:

$$(3.3) \qquad Y_-(J_+\gamma_0 u) = Y_-(J_+\gamma_0 u), \quad Y_+(\gamma_1 u) = Y_+JJ_+^{-1}(J_+\gamma_0 u).$$

Using also (2.16), we conclude that if $N(\eta,\xi') = (\eta - i|\xi'_\delta|)^{1/2}.(\eta + i|\xi'_\delta|)^{-1/2}$, and ϕ solves the operator equation

$$(3.4) \qquad \tilde{N}\phi = (Y_-\phi, Y_+F^{-1}N(\eta,\xi')F\phi) = (Y_-\phi, Y_+\psi),$$

operating from $W^{\theta,p}$ to $\Pi_\pm W^{\sigma,p}(R_\pm^n)$, $\sigma = k - 1 + \theta - 1/p$, then the solution u of the *MDP* $T_+Au = 0$, $J_+\gamma_0 u = \phi$ will solve the mixed problem (3.1) with the data $(0, Y_-\phi, Y_+\psi)$, (and conversely). Thus invertibility of (3.1) in $H^{k+\theta,p}$ is equivalent to the same property for the operator $\phi \to \tilde{N}\phi$ of (3.4) in $W^{\sigma,p}$.

Now $\tilde{N}$ is of the form (1.8). By lemma 1.1, it is invertible $H^{\sigma,p}$ if and only if $\tilde{N}_\sigma$ invertible for $(0,p)$, where in our case

$$(3.5) \qquad N_\sigma = \left(\frac{\eta - i|\xi'_\delta|}{\eta + i|\xi'_\delta|}\right)^{1/2+\sigma}.$$

This, in turn, happens if and only if the map $u \to (Y_-u, Y_+u)$ is invertible in $H^{\frac{1}{2}+\sigma,p}$. By Lemma (1.1) this is the case if and only if $-1 + 1/p < 1/2 + \sigma < 1/p$. Finally, from Remark 1.5 we deduce that this also is the condition of invertibility of $\tilde{N}$ in $W^{\sigma,p}$—and of the mixed problem in $H^{k+\theta,p}$. Expressing this condition in terms of $k + \theta$, we have

THEOREM 3.1. *The map 3.2 is invertible if and only if*

$$-1/2+2/p<k+\theta<1/2+2/p \tag{3.6}$$

the same result holds for any mixed problem of the type Dirichlet—oblique derivative, i.e. if γ_1 is replaced by $\gamma_0 B$, B given in (2.19).

Proof. We have to prove only the last assertion about the Dirichlet-oblique derivative problems. For that problem, the factor $N(\eta,\xi')$ is replaced by $M(\eta,\xi')$ of (2.21) which satisfies

$$M(\eta,\xi') = N(\eta,\xi')D(\eta,\xi') \tag{3.7}$$

where $D(\eta,\xi') = 1-(i\Sigma_j\alpha_j\xi_j)/(\eta^2+|\xi'|^2+\delta^2)$. Now $D(\eta,\xi')$ is never 0 and

$$\int_{\eta=-\infty}^{\eta=\infty} d_\eta[\operatorname{Arg} D(\eta,\xi')] = 0. \tag{3.8}$$

By the theory developed in [13] for operators of type (1.8)([1]) the relations (3.7), (3.8) imply that $\tilde{M}$ and $\tilde{N}$ are invertible for the same values of (s,p).

For each p (3.6) gives the range of invertibility (of length 1). This is also the range of regularization. A solution of a mixed problem in $H^{s,p}$ with a "better looking" data which allows (by existence) a more regular solution in $H^{t,p}$, which (by uniqueness) coincide with this other solution, provided s,t are in the range (3.6). The value $s=1/2+2/p$ is the barrier to higher regularization. However, the limited range (3.6) can be fully exploited even for general mixed problems as we propose to prove by the end of the paper.

We note that for $p<4/3$, $s=2$ is included in the range (3.6). Hence there is a unique (strong!) solution in $H^{2,p}$. For $p>4$, $s=0$ is in the range so that we can regularize weak L^p solutions. Both these facts will be used later. It is of interest to note that for $p=2$ the range (3.6) is $1/2<s<3/2$. Recall that the classical variational approach gives a unique solution in $H^{1,2}$.

REMARK 3.2. We can establish partial regularity of solutions of mixed problems for $-\Delta+\delta^2$ in the directions x_i, $i\geqq 2$, (thus excluding the directions $x_0=t$ and $x_1=y$). We observe that the invertibility results for pure and mixed problems for $T_+Au=T_+f$ were reduced to the corresponding problems for $T_+Au=0$ (by subtracting some $H^{2,p}$ solution of $T_+Au=T_+f$). In other words, we have used the decomposition

$$H_A^{s,p}(R_+^{n+1}) = H_{Au=0}^{s,p}(R_+^{n+1})+H^{2,p}(R_+^{n+1}), \qquad s\leqq 2, \tag{3.9}$$

([1]) The results of [13] were obtained for $M(\xi',\eta)$ homogeneous of order 0. But all the considerations remain true for factors $M(\xi',\eta,\delta)$ homogeneous in (ξ',η,δ), which we have here. One can see this by using for $H^{s,p}$ δ-norms (cf. Section 4) which have the same homogeneity, or by adding a new variable. We also note that in case M is a scalar (our case here) the theory of invertibility of $\tilde{M}$ is much easier than the general case.

where the meaning of the first summand is self-explanatory. (3.9) is easily seen to be a topological equivalence. The corresponding decomposition of the data of a boundary problem is given by the trace space cross $L^p(R^{n+1}_+)$. Now the re-relation $T_+Au = 0$ and all the boundary relations are invariant under translations in the directions x_i, $i \geqq 2$. I.e. $D_{x_i}u$, $D^2_{x_ix_j}u$, $i, j \geqq 2$. will satisfy the same relations and they will be in L^p if the corresponding derivatives of the boundary data of u belong to $W^{-1-1/p,p}$, or $\Pi_\pm W^{-1-1/p,p}(R^n_\pm)$ in the mixed case (where we assume $p > 4$ so that $(0, p)$ satisfies (3.6)). If desired, this partial regularity result (and higher ones) can be cast in a form of isomorphism theorems between suitable spaces.

REMARK 3.3. The values $s \equiv 1/p \pmod 1$ are exceptional in the trace theorem (Theorem 1.4) and were excluded up to now. These exceptional values can be avoided in the regularization process discussed in the next sections, and we shall omit mentioning them.

4. **Perturbed problems and local fundamental solution.** We start this section with an observation. In treating boundary problems concerning the operator $-\Delta + \delta^2$ we equip the spaces $H^{s,p}$ and the spaces derived from them (including trace spaces $W^{\sigma,p}$) with "δ-norms" which are homogeneous in the variables ξ (or τ, ξ) and δ together. Actually the operators (1.4), (1.5), (2.1), (2.2), which define the δ-norms, reappear in the solution of the boundary problems. The boundary operators and the domains (euclidean spaces or half spaces) are also homogeneous (although independent of δ). Hence we obtain

COROLLARY 4.1. *The norms of the isomorphisms established in Theorems* 2.3, 2.3 *bis*, 2.4, 2.5 *and* 3.1 *are independent of* δ.

Indeed one can start with the isomorphisms for a special value of δ, say 1, then pass to new coordinates by $X^* = \delta^{-1}X$. The equation $(-\Delta + 1)u = f$ goes over to

$$(-\Delta + \delta^2)u^* = \delta^2 f^* \tag{4.1}$$

Other relations change accordingly, with powers of δ reflecting the degree of homogeneity. Due to the parallel change in the norms of the spaces, the isomorphisms norms remain unchanged.

We shall continue to use δ-norms in this section and the next one. Estimation constants which are independent of δ will be denotes by K. For instance

$$\| D_j u \|_{s-1,p} \leqq K \| u \|_{s,p}. \tag{4.2}$$

Indeed, this amounts to a norm estimate for $F^{-1}\xi_j(\delta^2 + |\xi|^2)^{-1/2}F$, which is geneous in (ξ, δ).

Next we ask how the norm of a multiplication operator $u \to \phi(x)u$ in $H^{s,p}$ is effected by using a δ-norm for the space. The operator norm is denoted by

$\| \phi \|^{s,p}$. If $s = k$, a nonnegative integer and $\delta = 1$ then $\mathrm{Sup}_{|\alpha| \leqq k} \mathrm{Sup} | D^\alpha \phi |$ is the obvious estimate. Changing coordinates by $X^* = \delta^{-1} X$, we obtain for δ-norms

$$\| \phi \|^{k,p} \leqq K \operatorname*{Sup}_{|\alpha| \leqq k} \mathrm{Sup} | \delta^{-|\alpha|} D^\alpha \phi | \tag{4.3}$$

If $k - 1 < |t| \leqq k$ (then we set $k = t^+$), we can still use (4.3) as an estimate for $\| \phi \|^{t,p}$.

LEMMA 4.2. *Let $L = \Sigma a_\beta D^\beta$ be a differential operator of order m. Then as an operator from $H^{t+m,p}$ to $H^{t,p}$* (or between *similar W-spaces) its norm is estimated by $K \Sigma_p \| a_p \|^{t,p}$. The norm of $J_\delta L J_\delta^{-1}$ is estimated by*

$$K \Sigma_{\beta,j} (\| a_\beta \|^{t,p} + \| D_j a_\beta \|^{t,p})$$

For $J_\delta^2 L J_\delta^{-2}$ we have to add also $\| D_j^2 a_\beta \|^{t,p}$ to the summation.

Proof. The assertion about L clearly follows from (4.2). The operators $J_\delta^r L J_\delta^{-r}$ are pseudo-differential, with the same symbol as L and the assertions follow from Calderon results [16]. We treat here the case $r = 2$, which suffices for later purposes. Now $J_\delta^2 = \delta^2 + \Sigma D_j$ and

$$J_\delta^2 L J_\delta^{-2} = L + [J_\delta^2, L] J_\delta^{-2}.$$

J_δ^{-2} followed by the commutator $[J_\delta^2, L]$ is composed of terms

$$(D_j a_\beta D^\beta - a_\beta D^\beta D_j^2) J_\delta^{-2} = [2(D_j a_\beta) D^\beta] D_j J_\delta^{-2} + [(D_j^2 a_\beta) D^\beta] J_\delta^{-2} \tag{4.4}$$

Now J_δ^{-2} and $D_j J_\delta^{-2}$ have order 0 (i.e., map $H^{s,p}$ to itself), their symbols being standard L^p multipliers, homogeneous in (ξ, δ). The brackets in (4.4) are differential operators of order m at most with coefficients $D_j a_\beta$, $D_j^2 a_\beta$. Hence the result.

We turn now to boundary problems in a half sphere

$$S_r = \{(t,x) \mid t \geqq 0, \ |x|^2 + t^2 < r^2\}.$$

By $\partial S_r'$ we denote the *flat* part of ∂S_r, attaching + or − if we want the part in in $y \geqq 0$ or $y \leqq 0$.

$$A = (A, \gamma_0 B) \quad \text{or} \quad (A, Y_- \gamma_0 J_+, Y_+ \gamma_0 B) \tag{4.5}$$

is an abbreviation for a pure or a mixed boundary problem.

DEFINITION. An $H^{s,p}$ *[local] fundamental solution (fs)* [in *Sr*] for the mixed problem $Au = F = (f, \phi_-, \phi_+)$ is a bounded map

$$E : L^p(R^{n+1}) \times \Pi_\mp W^{\sigma,p}(R_\mp) \to H_A^{s,p}(R_+^{n+1}), \quad \sigma = s - 1 - 1/p \tag{4.6}$$

such that

$$AEF = F \ [\text{if Support } F \subset S_r] \tag{4.7}$$

$$EAu = u \text{ [if Support } u \subset S_r], \quad u \in H_A^{s,p} . \tag{4.8}$$

If $s \geqq 2$, *we replace* L^p *by* $H^{s-2,p}$ *in* (4.6) *and drop the subscript* A.

Thus E is a two-sided inverse of A [in the local case, when it is restricted to functions supported in S_r]. In the same way we define a [local] fs for a pure problem.

In Sections 2 and 3 we have constructed global fs for canonical problems, with $A = -\Delta + \delta^2$. We shall use in this section perturbation arguments to obtain local fs in S_r (r small) for general problems with smooth coefficients for which the principal part of A at the origin is $-\Delta$.

The coordinate transformation $X \to \delta^{-1}X$ maps S_δ onto S_1. Existence of a fs in S_δ for the original A is equivalent to the existence of a fs in S_1 for a transformed problem in which (cf. [5, Th. 10.4.1])

$$A_\delta = \delta^2 A(\delta X, \delta^{-1}D) = (-\Delta + \delta^2) + A^1 \tag{4.9}$$

$$B_\delta = \delta B(\delta X, \delta^{-1}D) = B_0 + B^1 \tag{4.10}$$

where B^0 is the principal part of B of the origin. If a_β, b_β, the coefficients of A and B, are in $C^{k+|\beta|}$ and a_β^1, b_β^1 are the coefficients of A^1, B^1 then

$$\delta^{-|\alpha|}D^\alpha a_\beta^1 = 0(\delta^{3-|\beta|}), \ |\alpha| < k + |\beta|, \ |\beta| > 0; \ = 0(\delta^2), \beta = 0; \tag{4.11}$$
$$= \delta^{2-|\beta|}o(1), \ |\alpha| = k + |\beta|$$

$$\delta^{-|\alpha|}D^\alpha b_\beta^1 = O(\delta^{2-|\beta|}), \quad |\alpha| < k + |\beta|; \tag{4.12}$$
$$= \delta^{1-|\beta|}o(1), \quad |\alpha| = k + |\beta|.$$

Notice that in (4.9) we have adjoined δ^2 to the principal part, but this does not effect the validity of (4.11).

We now extend A_δ, B_δ, defined by (4.9), (4.10) for $|X| \leqq 1$. Thus we set $A_\delta = -\Delta + \delta^2$ and $B_\delta = B_0$ for $|X| \geqq 2$. Then we choose some smooth interpolation to extend the definition to $1 < |X| < 2$. After the extension, the coefficients of the perturbation terms A_δ^1 B_δ^1 still satisfy (4.11) and (4.12), which imply by (4.3) and Lemma 4.2 that if k, the degree of smoothness of the coefficients, is large enough then *the operator norms of* A_δ^1 *and* B_δ^1 *(acting on* $H^{s,p}$*) tend to* 0 *with* δ. If s increases, so should k, but for $s < 3$ it suffices to take $k = 1$. Now the problems associated with $-\Delta + \delta^2$ and B_0 are invertible (the mixed one if (3.6) is satisfied) *with operator norm idenpendent of* δ. Thus we proved part (a) of the following

THEOREM 4.3(a). *Let* $2 \leqq s$ *and* δ *sufficiently small. There exist* $H^{s,p}$ *local* fs *in* S_δ *for the problems considered above (for the mixed one,* (s,p) *should also satisfy* (3.6)).

(b) *The assertion in* (a) *is true for problems determined by* $J_1^{-2}AJ_1^2$ *(instead of* A*) and* B. *Only the smoothness conditions on* A *increase by* 2.

(c) *The assertion in* (b) *implies the existence of an* $H_A^{s,p}$ *local fs in* S_δ, $0 \leqq s < 2$, (*for mixed problems, if* (s,p) *satisfies* (3.6)).

Proof. For part (b), we change X to $\delta^{-1}X$ as above. Then we have to construct a *fs* in S_1 for

$$(4.13) \qquad A_\delta^* = \delta^2 J_\delta^2 A(\delta X, \delta^{-1}D) J_\delta^{-2} = -\Delta + \delta^2 + J_\delta^2 A^1 J_\delta^{-2}$$

and B_δ of (4.10). We extend A_δ^* and B_δ to R_+^{n+1} as in part (a). To estimate the perturbation term in (4.13) we again use Lemma 4.2 and the estimate (4.11).

For part (c) we cannot use a straightforward perturbation argument since the underlying spaces $H_A^{s,p}$ change with A if $s < 2$. Instead, we argue as follows. A_δ and A_δ^* are elliptic for small δ and by parts (a), (b) their Dirichlet problems are invertible in $H^{s,p}$ $s \geqq 2$. Let G_δ, G_δ^* denote the *fs* for the corresponding modified Dirichlet problems. Let $G_\delta\phi = G_\delta(0,\phi)$, $G_\delta^*\phi = G_\delta^*(0,\phi)$. Obviously $G_\delta^* = J_\delta^{-2} G_\delta J_\delta^2$.

By Remark 2.6 $\gamma_0 J_+$ and $\gamma_0 B_\delta$ can be defined in $H_A^{s,p}$ for $0 \leqq s < 2$ and the MDP is invertible, i.e., G_δ exists. Moreover, for $\delta \to 0$ the norms of (A_δ, γ_0) and its inverse G_δ approach those of $(-\Delta + \delta^2, \gamma_0)$ and its inverse (as for $s \geqq 2$). Hence these norms are estimated independently of δ for small δ. For any $s \geqq 0$ the operator

$$(4.14) \qquad M_\delta : \phi \to \gamma_0 B_\delta G_\delta \phi \text{ acts in } W^{\sigma,p}(R^n), \qquad \sigma = s - 1 - 1/p,$$

and gives the relation between the boundary data $\gamma_0 J_+ u$ and $\gamma_0 B_\delta u$ for solutions of $T_+ A u = 0$. The corresponding operator for the unperturbed case (with $-\Delta + \delta^2$ and B_0) was given in (2.20), (2.21) and is denoted now by M_0. It is known to be invertible in $W^{\sigma,p}$ (with norm independent of δ). We claim that

$$(4.15) \qquad \text{The operator norm } \| M_\delta - M_0 \|^{s,p} \text{ tends to } 0 \text{ with } \delta.$$

We have $M_\delta = \gamma_0 B^0 G_\delta + \gamma_0 B^1 G_\delta$. In the second term G_δ and γ_0 have norm independent of δ, while the norm of B^1 tends to 0 as in part (a). Thus we may assume $B = B^0$. Then

$$(4.16) \qquad \| M_\delta - M_0 \|^{\sigma,p} = \operatorname*{Sup}_\phi \frac{\|(M_\delta - M_0)\phi\|_{\sigma,p}}{\|\phi\|_{\sigma,p}} = \operatorname*{Sup}_\psi \frac{\|(M_\delta - M_0)J_\delta^2\psi\|_{\sigma,p}}{\|J^2\psi\|_{\sigma,p}}$$
$$= \operatorname*{Sup}_\psi \frac{\|J_\delta^{-2}(M_\delta - M_0)J_\delta^2\psi\|_{\sigma+2,p}}{\|\psi\|_{\sigma+2,p}}.$$

Let $M_\delta^* = \gamma_0 B^0 G_\delta^* = J_\delta^{-2}\gamma_0 B^0 G_\delta J_\delta^2 = J_\delta^{-2} M_\delta J_\delta^2$ and then $M_\delta - M_0^* = J_\delta^{-2}(M_\delta - M_0)J_\delta^2$. (4.16) gives its operators norm in $W^{\sigma+2,p}$ the smallness of which as $\delta \to 0$ is equivalent to the smallness in norm of $J_\delta^2 A^1 J_\delta^{-2}$ acting on $H^{s+2,p}$. This last fact was proved in part (b), in constructing a *fs* for problems associated with A_δ (notice that $s + 2 \geqq 2$). Thus (4.15) is proved.

Cleary (4.15) implies that M_δ is invertible for small δ, hence there is a local fs in $H_A^{s,p}$ for the pure boundary problem $(A, \gamma_0 B)$. Also (4.15) implies that $\| \tilde{M} - \tilde{M}_0 \|^{\sigma,p} \to 0$, hence, for small δ, $M = (Y_-, Y_+ M_\delta)$ is invertible whenever $\tilde{M}_0$ is. This proves the existence of a local fs for the mixed problem in case (s,p) satisfy (3.6). This concludes the proof of Theorem 4.3. It is easily checked that the smoothness condition—the coefficients a_β and b_β of A and B belong to $C^{1+|\beta|}$—is enough.

5. **Regularity in the general case.** In this section we establish regularity of a weak solution of a general mixed problem. We consider:

I. A closed domain $\bar{\Omega} \subset R^{n+1}$ which is a C^3-manifold with boundary $\partial\Omega$. A C^3 $(n-1)$-dimensional Γ divides $\partial\Omega$ into $\partial\Omega^-$ and $\partial\Omega^+$.

II. $A = \Sigma a_\beta(X) D^\beta$, a second order elliptic operator with real coefficients $a_\beta \in C^{1+|\beta|}(\bar{\Omega})$.

III. $B = \Sigma b_\beta(X) D^\beta$, a first order operator with real coefficients which covers A of each point of $\partial\Omega$. Also $b_\beta(x) \in C^{1+|\beta|}(\bar{\Omega})$.

A function $u \in L^p(\Omega)$ is a solution of the mixed problem $u = F$, $F = (f, \phi, \psi)$, if

$$(5.1) \qquad Au = f \text{ in } \Omega, \ \gamma_0 u = \phi \text{ on } \partial\Omega^-, \ \gamma_0 Bu = \psi \text{ on } \partial\Omega^+$$

f is assumed to be in some (or all) L^p, $p > 1$. Then (cf. Remark 2.6) $\gamma_0 u$, $\gamma_0 Bu$ are well defined and belong to $W^{-1/p,p}(\partial\Omega)$ and $W^{-1-1/p,p}(\partial\Omega)$ respectively. We assume at least this much (but usually more smoothness) of ϕ and ψ so that (5.1) makes sense.

If u satisfies (5.1) only in a certain neighborhood of $X_0 \in \Gamma$, we call it a local solution. In this case, I–II–III need be satisfied only in that neighborhood. We remark that (5.1) coincides with the customary notion of weak solution (or local solution). Cf. (5.5).

We can perform (in steps) an invertible C^2-transformation of coordinates which will:

(a) Take a local solution defined near $X_0 \in \Gamma$ to a local solution of a mixed problem in a half sphere S_r (X_0 goes to the origin and Γ us contained in $y = 0$, $t = 0$).

(b) Get rid of the mixed terms $D^2_{tx_j}$, $1 \leqq j \leqq n$ in the operator A. This is obtained by a transformation of the form $t^* = t$, $x^* = x^*(t,x)$ with $x^*(0,x) = x$, i.e., the transformation is the identity on the plane $t = 0$. Cf. [2, page 90] for details.

(c) Get rid of the mixed terms $D^2_{yx_j}$, $2 \leqq j \leqq n$, on the plane $t = 0$. This is done as in (b), but for the variables $y, x_2, \cdots, x_n$. After this, we have in S_r $a_{yx_j}(t,y,x') = O(t)$, $t \to 0$. Here a_{yx_j} is the coefficient of $D^2_{yx_j}$ in A.

(d) Set the principal part of A of the origin equal to $-\Delta$. For this we divide by minus a_{tt} and use an affine transformation on $y, x_2, \cdots, x_n$. (Performing a further rotation if necessary, we may assume the plane $y = 0$ is preserved.)

The boundary operator B in the transformed problem still covers the transformed A. The smoothness properties of the coefficients, the data and the solution itself are clearly preserved by the coordinate transformation and its inverse. Hence it suffices to prove the regularity results for a local solution of a mixed problem $\mathbf{A} = (A, Y_-\gamma_0 J_+, Y_+\gamma_0 B)$ in a half sphere S_r, and we may assume that (b), (c), (d) were done.

LEMMA 5.1. *Let* $1 < q < 4$, $f \in L^q$, $\phi_\pm \in W^{1/q,q}(R^n_\pm)$ *and let* $u \in H^{1,p}$, $p < q$, *be a local solution of* $\mathbf{A}u = \mathbf{F} = (f, \phi_-, \phi_+)$. *Then actually* $u \in H^{1,q}$.

Proof. Here (and later) it suffices to prove the assertion in some S_δ. For $1 < p < 4$ we verify that (s,p) satisfies (3.6) for s in an interval around $s = 1$. Thus for some δ there is a $H^{s,p}$ *fs* in S_δ. Let ζ be a C^∞ function which is 0 for $|X| \geqq \delta$ and 1 for $|X| \leqq \frac{1}{2}\delta$. Then

$$A\zeta u = \zeta A u + A' u \in L^p \tag{5.2}$$

since A' is a 1st order operator. The boundary data of ζu are in $W^{-1/p,p}(R^n_\pm)$ (near the origin) as one easily verifies; hence, applying the *fs* $\mathbf{E}$ we get that in S_δ

$$\zeta u = \mathbf{E}\mathbf{A}(\zeta u) \in H^{s,p} \text{ for any } s < \frac{1}{2} + \frac{2}{p}. \tag{5.3}$$

The same is true for u in $S_{\frac{1}{2}\delta}$. In particular we can choose $s = 1 + 1/p < 1/2 + 2/p$. By the Sobolev (fractional) embedding theorem [9], we obtain $u \in H^{1,p_1}$ with $p_1^{-1} = p^{-1}\left(1 - \dfrac{1}{n+1}\right)$. If $p_1 > q$ we have finished. Otherwise we repeat the process m times until $p_m^{-1} = p^{-1}\left(1 - \dfrac{1}{n+1}\right)^m < q^{-1}$. Then $u \in H^{1,p_m} \subset H^{1,q}$.

If the weak solution u is just in L^p (instead of $H^{1,p}$) it is not evident that the term $A'u$ in (5.2) is in L^p. To prove this is so, we need the following "partial regularity" theorem, which will be validated later:

THEOREM 5.2. *Let* $p > 4$, $f \in L^p$, $\phi_\pm \in W^{-1/p,p}(R^n_\pm)$. *Let* $u \in L^p$ *be a local solution of* $\mathbf{A}u = \mathbf{F}$. *Then* $D_j u \in L^p$, $j \geqq 2$.

THEOREM 5.3. *Let* I, II *and* III *be satisfied and let* $p > 4$. *Assume* $f \in L^p$, $\phi_\pm \in W^{-1/p,p}(R^n_\pm)$ *and let* $u \in H^{1,q}$, $1 < q < p$ *or* $u \in L^{p_0}$, $p_0 > 4$. *Let* u *be a local solution of* $\mathbf{A}u = \mathbf{F}$. *Then* $u \in H^{s,p}$ *for any* $s < \dfrac{1}{2} + \dfrac{2}{p}$.

Proof. If $u \in H^{1,q}$ then by Lemma 5.1 it is in $H^{1,4-\varepsilon}$ for any $\varepsilon > 0$, hence in L^{p_0} for some $p_0 > 4$. Next we observe that it is enough to prove the theorem for $p_0 = p$ because if $p_0 < p$ we first obtain $u \in H^{s,p_0}$ and use the Sobolev theorem to obtain $u \in L^{p_1}$, $p_1 > p_0$. Repeating the argument several times we get $u \in L^p$.

Now the choice of s and p ensures that $(p,0)$ and (s,p) satisfy (2.6), and by Theorem 4.3(c) there is a local fs in S_δ. Let $\zeta(X) = \Pi_j\zeta(x_j)$ where $\zeta(z)$ is a C^∞ function of one variable supported in $z^2 < \delta^2(n+1)^{-1}$ (so that $\zeta(X)$ is supported in S_δ), and is 1 in $z^2 < \delta^2(4(n+1))^{-1}$. We claim that $A(\zeta u) \in L^p$. The only terms in doubt are $a_{ij}D_i\zeta D_j u$ coming from the terms $a_{ij}D_{ij}^2$ in A. For $j \geqq 2$ they are in L^p by Theorem 5.2. For $i = j = 0$ $D_0\zeta = D_t\zeta$ vanishes near $t = 0$ and for $i = j = 1$ $D_1\zeta = D_y\zeta$ vanishes near $y = 0$. However outside a neighborhood of $t = 0$, $y = 0$ we know $D_j u \in L^p$ by the (local) interior regularity or boundary regularity for pure problems. In the mixed terms $j = 0$ $i \geqq 1$ the coefficient a_{i0} vanishes because of the special form of A described in (b) above, while by (c) $a_{i1} = 0(t)$. Thus it suffices to know that $tD_j u \in L^p$. This (and also $t^2D_{ij}u \in L$) follows quite easily from the a-priori estimates for the Dirichlet problem (one multiplies u by a function ψ which vanishes near $t = 0$ and estimates ψu).

COROLLARY 5.4. *If $u \in L^p$, $\phi_\pm \in W^{-1/p,p}(R^n_\pm)$ for all p then any local solution of a mixed problem which is in L, $q > 4$ is in Hölder class α for any $\alpha < \frac{1}{2}$.*

Indeed it is in $H^{1/2,p}$ for any p, hence in C^α by the Sobolev embedding theorem (fractional form [9]).

We return now to Theorem 5.2. After subtracting a (smooth) solution of a Dirichlet problem, we may assume $Au = f$, $Y_-\gamma_0 u = 0$ $Y_+\gamma_0 Bu = \phi_+$. Let $(A^*, \gamma_0, \gamma_0 B^*)$ be the adjoint mixed problem. It is uniquely determined and I, II, III are satisfied for it. Let V_{mixed} be the class of smooth functions which vanish outside S_r and near its curved boundary, while on the flat part they satisfy homogeneous adjoint boundary conditions.

$$\gamma_0 v = 0 \quad \text{for } y \leqq 0, \ \gamma_0 B^* v \text{ for } y \geqq 0. \tag{5.4}$$

Using Green's formula we obtain for $v \in V_{mixed}$

$$\begin{aligned} \int uA^*v dX &= \int f \cdot v dX - \int_{y>0} \gamma_0(Bu \cdot v) dx \\ &= (f, v) - \langle Y_+\phi, \gamma_0 v \rangle \end{aligned} \tag{5.5}$$

where $Y_+\phi \in W^{-1/p,p}(R^n_+)$ and $\langle\ ,\ \rangle$ is the duality between this space and $W_+^{1-1/p',p'}$. (If $Y_-\gamma_0 u = Y_-\psi \neq 0$ we get an additional term $\langle Y_-\psi, \gamma_0 B^* v\rangle$ on the right side of (5.5), and this relation is the customary definition of a local weak solution.) From (5.5) we get

$$\begin{aligned} |(u, A^*v)| &\leqq \|f\|_{0,p}\|v\|_{0,p'} + \|\phi\|_{-1/p,p}\|\gamma_0 v\|_{1-1/p',p'} \\ &\leqq K\|v\|_{1,p'}, \quad v \in V_{mixed}. \end{aligned} \tag{5.6}$$

The analogous inequality for $V_{\text{Dirichlet}}$ implies $u \in H^{1,p}$. This is proved in [1] where it is first established that the derivatives $D_j u \in L^p$, $j \geqq 1$ [1, Lemma 5.2].

The translation argument there carries over to our case for the derivatives $D_j u$, $j \geqq 2$ (excluding the t and y directions). Indeed the main tool there is the local *fs* of the adjoint problem in the space $H^{2,p'}$. [1, Lemma 4, 2]. Using the condition (3.6) we see that $H^{2,p'}$ local *fs* for the mixed case exists of $p' < 4/3$, i.e., $p > 4$. We notice that the proof referred to in [1] can be extended with a minor change, to V_{mixed} or V_B (B a pure boundary condition) although the spaces are not invariant under translations in the x_i directions, but we omit the details.

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